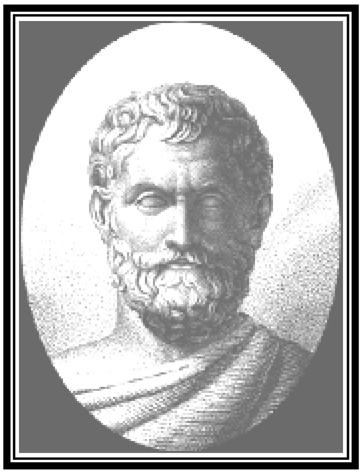
Chapter

4

Heights and Distances

CONTENTS



Thales

The name of 'Thales' (about 600 B.C) is invariably associated with height and distance problems. He is credited with the determination of the height of a great Pyramid in Egypt by measuring shadows of pyramid and an auxiliary staff (or gnomon) of known height and comparing the ratios.

$$\frac{H}{S} = \frac{h}{s} = \tan \ \textit{(sun's altitude)}$$

Thales is also said to have Calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity are also found in ancient indian works.

Heights and Distances

This chapter deals with the applications of trigonometry to practical situations concerning measurement of heights and distances which are otherwise not directly measurable By the use of trigonometry we can measure the following :

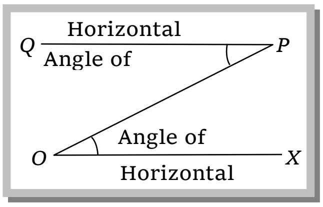
(i) Height of tower or temple (ii) Breadth of river

(iii) Distance between inaccessible points (iv) Angle of vision etc.

We need to first define certain terms and state some properties before applying the principles of trigonometry.

4.1 Some Terminology Related to Heights and Distances

(1) **Angle of elevation and depression:** Let O and P be two points such that P is at higher level than O. Let PQ, OX be horizontal lines through P and O, respectively. If an observer (or eye) is at O and the object is at P, then $\angle XOP$ is called the angle of elevation of P as seen from O. This angle is also called the angular height of P from O.



If an observer (or eye) is at P and the object is at O, then $\angle QPO$ is called the angle of depression of O as seen from P.

(2) **Method of solving a problem of heights and distances**

(i) Draw the figure neatly showing all angles and distances as far as possible.

(ii) Always remember that if a line is perpendicular to a plane then it is perpendicular to every line in that plane.

(iii) In the problems of heights and distances we come across a right angled triangle in which one (acute) angle and a side is given. Then to find the remaining sides, use trigonometrical ratios in which known (given) side is used, *i.e.*, use the formula.

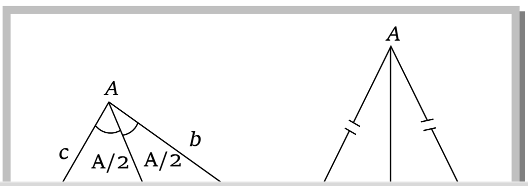
(iv) In any triangle other than right angled triangle, we can use 'the sine rule'.

i.e., formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, or cosine formula *i.e.*, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ etc.

(v) Find the length of a particular side from two different triangles containing that side common and then equate the two values thus obtained.

(3) **Geometrical properties and formulae for a triangle**

(i) In a triangle the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle. $\frac{BD}{DC} = \frac{c}{b}$.

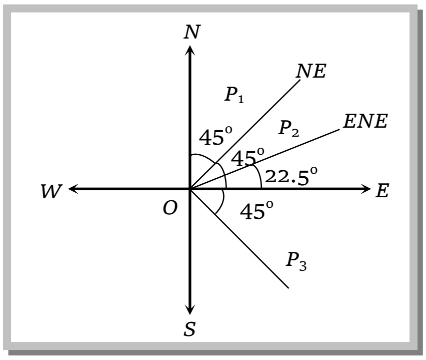


(ii) In an isosceles triangle the median is perpendicular to the base *i.e.,* $AD \perp BC$.

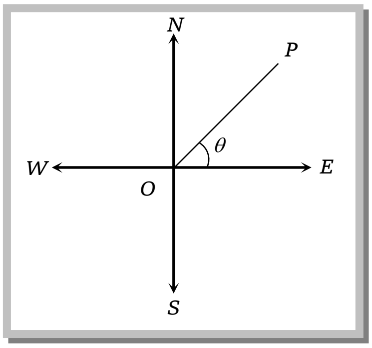
(iii) In similar triangles the corresponding sides are proportional.

(iv) The exterior angle is equal to sum of interior opposite angles.

(4) **North-east:** North-east means equally inclined to north and east, south-east means equally inclined to south and east. *ENE* means equally inclined to east and north-east.



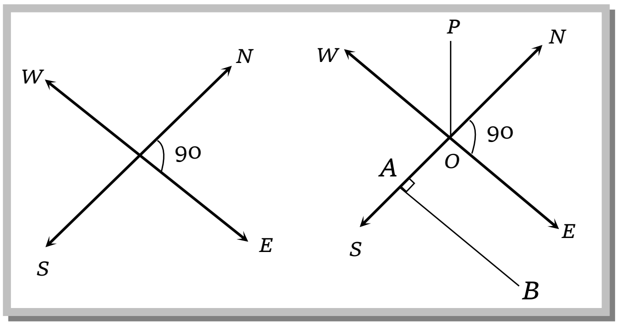
(5) **Bearing :** In the figure, if the observer and the object *i.e., O* and *P* be on the same level then bearing is defined. To measure the 'Bearing', the four standard directions East, West, North and South are taken as the cardinal directions.



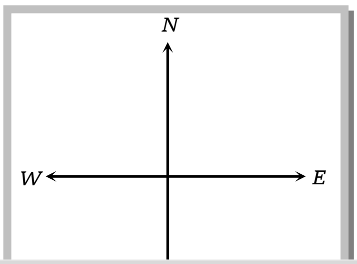
Angle between the line of observation *i.e., OP* and any one standard direction– east, west, north or south is measured.

Thus, $\angle POE = \theta$ is called the bearing of point *P* with respect to *O* measured from east to north. In other words the bearing of *P* as seen from *O* is the direction in which *P* is seen from *O*.

(6) **Problem on two dimensions :** If the actual figure is located in one plane, the problem is of two dimensions. For direction in two dimensional figures, cross vertically as shown in the figure.



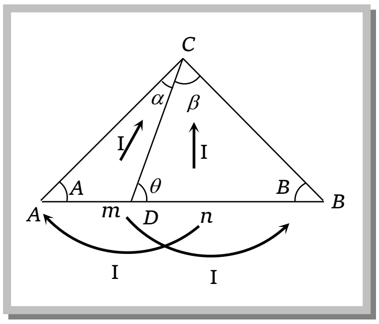
(7) **Problems on three dimensions :** If total actual figure is located in more than one plane, the problem will be of three dimensions. For



direction in three dimensional figures, cross obliquely as shown. Clearly this oblique cross represents the horizontal plane.

If OP be a vertical tower perpendicular to the plane then it will be represented like the figure, clearly $\angle POA = 90^o$. If the observer at A moves in east direction. We draw a line AB parallel to east to represent this movement. Clearly $\angle OAB = 90^o$ (angle between north and east).

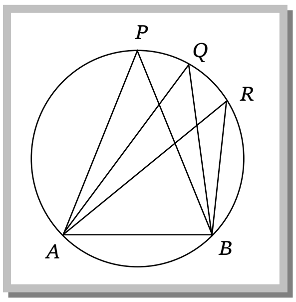
(8) ***m-n* cot theorem of trigonometry:** $(m+n)\ \cot\theta\ = m\cot\alpha - n\cot\beta = n\cot A - m\cot B$ (θ on the right)



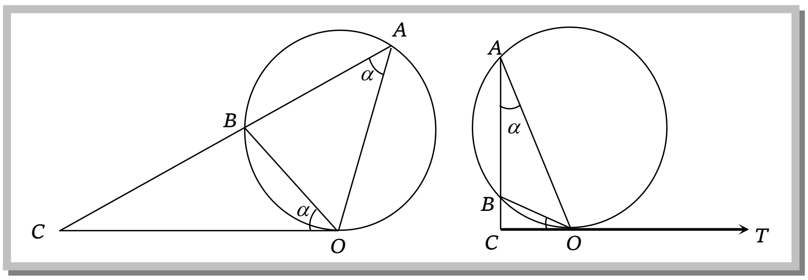
Note : ❑ If θ is on the left then angle in the right is $\pi - \theta$ and $\cot(\pi-\theta) = -\cot\theta$. Hence in this case m- n theorem becomes $-(m+n)\cot\theta = m\cot\alpha - n\cot\beta\ = n\cot A - m\cot B$ (θ on the left).

4.2 Some Properties Related to Circle

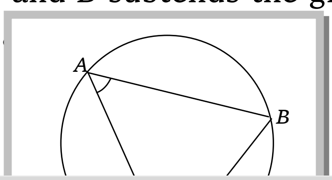
(1) Angles in the same segment of a circle are equal *i.e.*, $\angle APB = \angle AQB = \angle ARB$.



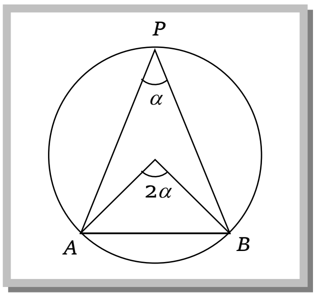
(2) Angles in the alternate segments of a circle are equal.



(3) If the line joining two points A and B subtends the greatest angle α at a point P then the circle, will touch the straight line XX' .

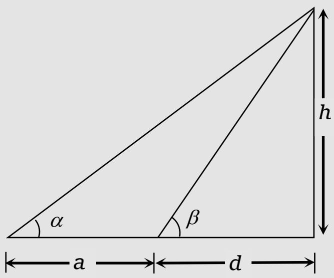


(4)The angle subtended by any chord at the centre is twice the angle subtended by the same on any point on the circumference of the circle.



4.3 Some Important Results

(1)

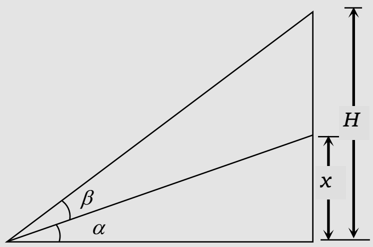


$$a = h(\cot\alpha - \cot\beta) = \frac{h\sin(\beta-\alpha)}{\sin\alpha.\sin\beta}$$

$\therefore h = a\sin\alpha\sin\beta\,\text{cosec}\,(\beta-\alpha)$ and

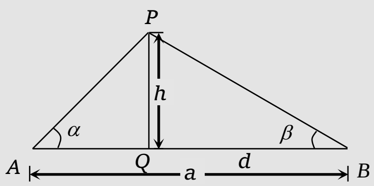
$d = h\cot\beta = a\sin\alpha.\cos\beta.\text{cosec}(\beta-\alpha)$

(2)



$$H = x\cot\alpha\tan(\alpha+\beta)$$

(3)

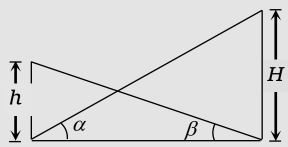


$a = h(\cot\alpha + \cot\beta)$, where by

$h = a\sin\alpha.\sin\beta.\text{cosec}(\alpha+\beta)$ and

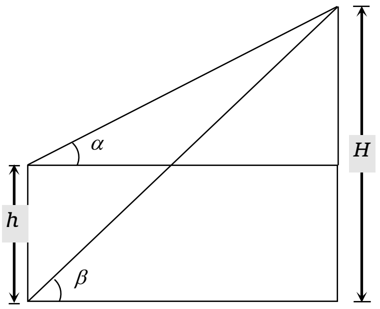
$d = h\cot\beta = a\sin\alpha.\cos\beta.\text{cosec}\,(\alpha+\beta)$

(4)



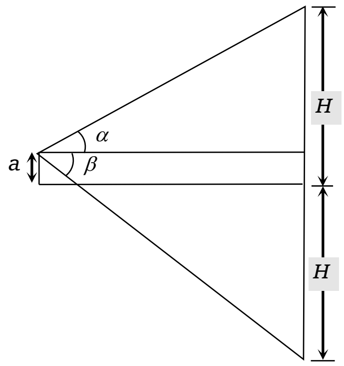
$$H = \frac{h\cot\beta}{\cot\alpha}$$

(5)



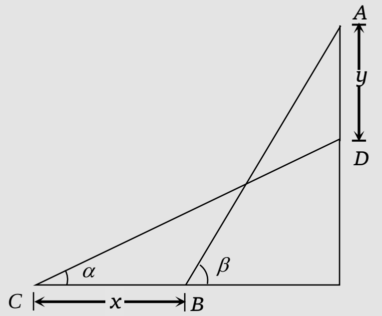
$h = \dfrac{H \sin(\beta - \alpha)}{\cos\alpha \sin\beta}$ or $H = \dfrac{h \cot\alpha}{\cot\alpha - \cot\beta}$

(6)



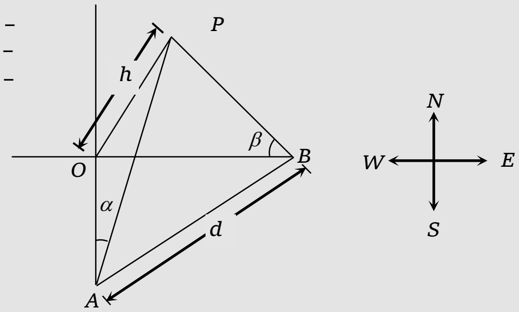
$H = \dfrac{a \sin(\alpha + \beta)}{\sin(\beta - \alpha)}$

(7)



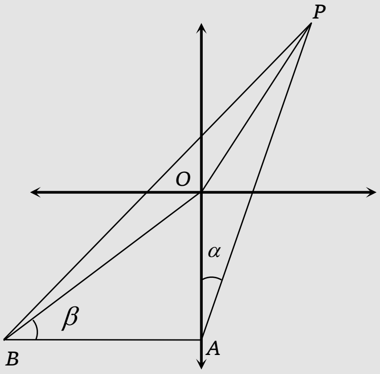
$AB = CD$. Then, $x = y \tan\left(\dfrac{\alpha + \beta}{2}\right)$

(8) OP –
A –
B –



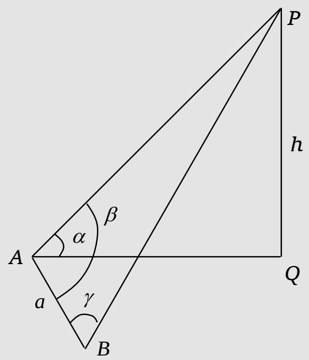
$h = \dfrac{d}{\sqrt{\cot^2\beta + \cot^2\alpha}}$

(9)



$h = \dfrac{AB}{\sqrt{\cot^2\beta - \cot^2\alpha}}$

(10)



$h = AP \sin\alpha = a \sin\alpha . \sin\gamma . \text{cosec}\,(\beta - \gamma)$ and

if $AQ = d$, then

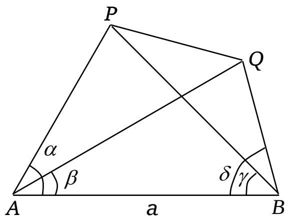
$d = AP \cos\alpha = a \cos\alpha . \sin\gamma . \text{cosec}\,(\beta - \gamma)$

(11) $AP = a\sin\gamma.\text{cosec}(\alpha-\gamma)$

$AQ = a\sin\delta.\text{cosec}(\beta-\delta)$

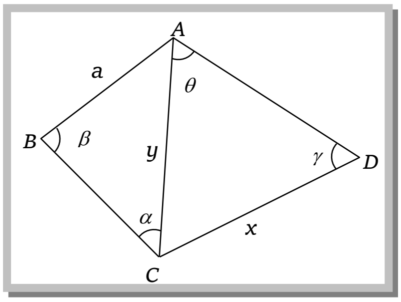
and apply,

$PQ^2 = AP^2 + AQ^2 - 2AP.AQ\cos\theta$



Important Tips

☞ *In the application of sine rule, the following point be noted. We are given one side a and some other side x is to be found. Both these are in different triangles. We choose a common side y of these triangles. Then apply sine rule for a and y in one triangle and for x and y for the other triangle and eliminate y. Thus, we will get unknown side x in terms of a. In the adjoining figure a is known side of Δ ABC and x is unknown is side of triangle ACD. The common side of these triangle is AC = y (say) Now apply sine rule*



$\therefore \quad \frac{a}{\sin\alpha} = \frac{y}{\sin\beta}$ .... *(i)* and $\frac{x}{\sin\theta} = \frac{y}{\sin\gamma}$ ....*(ii)*

Dividing (ii) by (i) we get, $\frac{x\sin\alpha}{a\sin\theta} = \frac{\sin\beta}{\sin\gamma}$; $\therefore \quad x = \frac{a\sin\beta\sin\theta}{\sin\alpha\sin\gamma}$

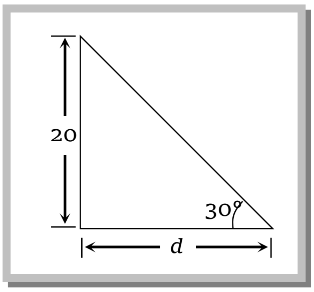
4.4 Miscellaneous Examples

Example: 1 The angle of elevation of a tower at a point distant *d metres* from its base is 30^o. If the tower is 20 meters high, then the value of *d* is

(a) $10\sqrt{3}\,m$ (b) $\frac{20}{\sqrt{3}}\,m$ (c) $20\sqrt{3}\,m$ (d) $10\,m$

Solution: (c) $\frac{d}{20} = \cot 30^o$

$d = 20\cot 30^o = 20\sqrt{3}m$.



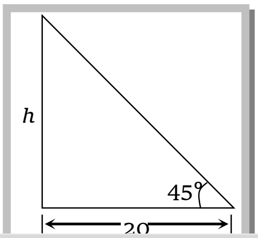
Example: 2 The angle of elevation of the top of a tower from a point 20 *meters* away from its base is 45^o. The height of the tower is

[MP PET 1984, 1989]

(a) 10 *m* (b) 20 *m* (c) 40 *m* (d) $20\sqrt{3}\,m$

Solution: (b) Let height of the tower be *h*.

$\frac{h}{20} = \tan 45^o$



$h = 20m$.

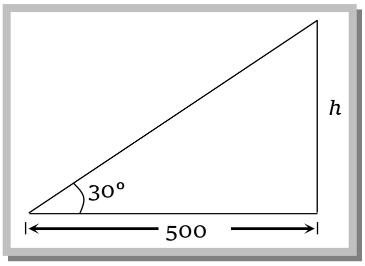
Example: 3 If the angle of elevation of the top of a tower at a distance 500 *m* from its foot is 30^o, then height of the tower is

[Kerala (Engg.) 2002]

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{500}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\frac{1}{500}$

Solution: (b) Let the height be *h*

$\therefore \tan 30^o = \frac{h}{500} \Rightarrow h = \frac{500}{\sqrt{3}}$.



Example: 4 A person standing on the bank of a river finds that the angle of elevation of the top of a tower on the opposite bank is 45^o. Then which of the following statements is correct

(a) Breadth of the river is twice the height of the tower

(b) Breadth of the river and the height of the tower are the same

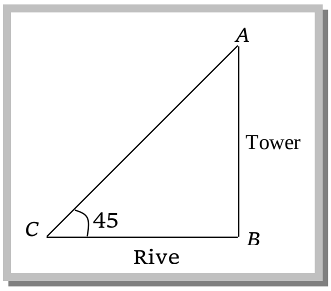
(c) Breadth of the river is half of the height of the tower

(d) None of these

Solution: (b) *AB* is tower and *BC* is river.

From $\Delta ABC, \frac{AB}{BC} = \tan 45^o$ or $AB = BC$

Height of tower = Breadth of river.



Example: 5 A ladder 5 *metre* long leans against a vertical wall. The bottom of the ladder is 3 *metre* from the wall. If the bottom of the ladder is pulled 1 *metre* farther from the wall, how much does the top of the ladder slide down the wall **[AMU 2000]** **[AMU 2000]**

(a) 1 *m*

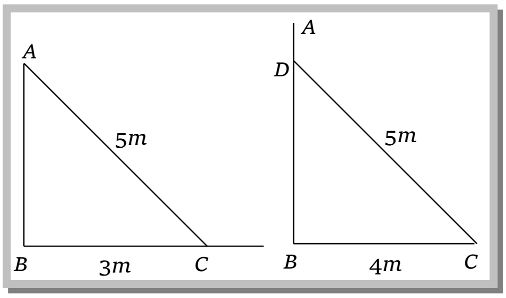
(b) 7 *m*

(c) 2 *m*

(d) None of these

Solution: (a) $AB = 4m \Rightarrow BD = 3m$

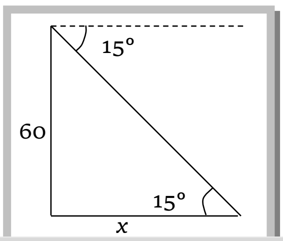
$\therefore AD = 4 - 3 = 1m$.



Example: 6 From the top of a light house 60 *metre* high with its base at the sea level the angle of depression of a boat is 15°. The distance of the boat from the foot of the light house is **[MP PET 2001, 1994; IIT 1983; U**

(a) $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) 60$ *metre* (b) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) 60$ *metre* (c) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$ *metre* (d) None of these

Solution: (b) Required distance = $60 \cot 15^o$



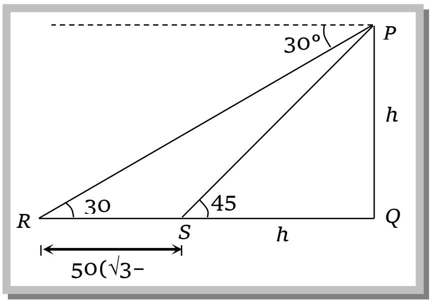
$$= 60\left[\frac{\sqrt{3}+1}{\sqrt{3}-1}\right] \text{ metre.}$$

Example: 7 A person observes the angle of deviation of a building as 30^o. The person proceeds towards the building with a speed of $25(\sqrt{3}-1) m/hour$. After 2 *hours*, he observes the angle of elevation as 45^o. The height of the building (in *metres*) is

[UPSEAT 2003]

(a) 100 (b) 50 (c) $50(\sqrt{3}+1)$ (d) $50(\sqrt{3}-1)$

Solution: (b) In ΔPQR, $\tan 30^o = \frac{PQ}{QR}$



$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{50(\sqrt{3}-1)+h}$

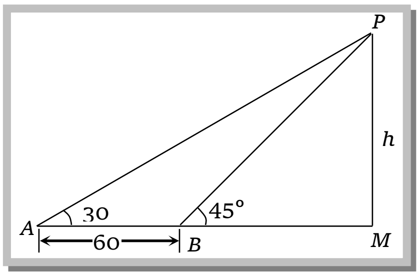
$\Rightarrow \sqrt{3}h = 50(\sqrt{3}-1)+h$

$\Rightarrow (\sqrt{3}-1)h = 50(\sqrt{3}-1) \Rightarrow h = 50$ *metre.*

Example: 8 The shadow of a tower standing on a level ground is found to be 60 *m* longer when the sun's altitude is 30^o than when it is 45^o. The height of the tower is

(a) 60 *m* (b) 30 *m* (c) $60\sqrt{3}m$ (d) $30(\sqrt{3}+1)m$

Solution: (d) $\because AB = AM - BM \Rightarrow \frac{AB}{h} = \frac{AM}{h} - \frac{BM}{h}$



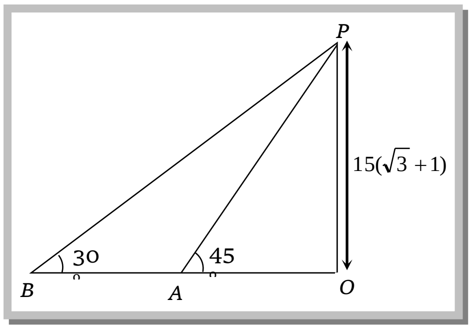
$\frac{AB}{h} = \cot 30^o - \cot 45^o \Rightarrow h = \frac{60}{\sqrt{3}-1} = \frac{60(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$\Rightarrow h = \frac{60(\sqrt{3}+1)}{3-1} \Rightarrow h = 30(\sqrt{3}+1)m$.

Example: 9 A person is standing on a tower of height $15(\sqrt{3}+1)m$ and observing a car coming towards the tower. He observed that angle of depression changes from 30^o to 45^o in 3 sec. What is the speed of the car

(a) 36 *km/hr* (b) 72 *km/hr* (c) 18 *km/hr* (d) 30 *km/hr*

Solution: (a) $AB = OP[\cot\alpha - \cot\beta]$, where $\alpha = 30°, \beta = 45°$



$\Rightarrow AB = 15(\sqrt{3}+1)(\sqrt{3}-1) = 15(3-1) = 30$ *metre.*

Speed = $\frac{\text{Distance}}{\text{time}} = \frac{30}{3} = 10 m/\text{sec}$

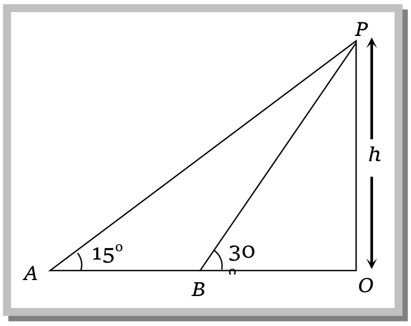
$= 10 \times \frac{18}{5} km/hr$

= 36 *km/hr.*

Example: 10 The angle of elevation of the top of a pillar at any point *A* on the ground is 15°. On walking 40 *metre* towards the pillar, the angle becomes 30°. The height of the pillar is

(a) 40 *metre* (b) 20 *metre* (c) $20\sqrt{3}$ *metre* (d) $\frac{40}{3}\sqrt{3}$ *metre*

Solution: (b) $AB = OP[\cot\alpha - \cot\beta] \Rightarrow 40 = h[\cot 15^o - \cot 30^o]$



$h = \frac{40}{\cot 15^o - \cot 30^o}$

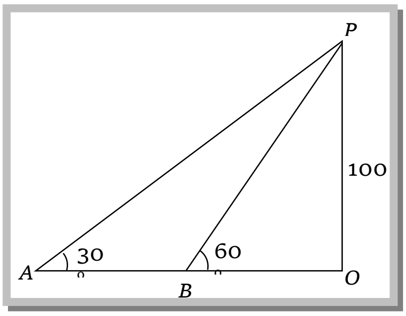
$h = \frac{40}{\frac{1}{2-\sqrt{3}} - \sqrt{3}} = \frac{40(2-\sqrt{3})}{4-2\sqrt{3}}$

$h = 20$ *metre* .

Example: 11 A man from the top of a 100 *metre* high tower looks a car moving towards the tower at an angle of depression of 30°. After some time, the angle of depression becomes 60°. The distance (in *metre*) travelled by the car during this time is **[IIT Screening 2001]**

(a) $100\sqrt{3}$ (b) $\frac{200\sqrt{3}}{3}$ (c) $\frac{100\sqrt{3}}{3}$ (d) $200\sqrt{3}$

Solution: (b) $d = AB = OA - OB$



$= 100[\cot 30^o - \cot 60^o]$

$= 100\left[\sqrt{3} - \frac{1}{\sqrt{3}}\right] = \frac{200\sqrt{3}}{3}$ *metre.*

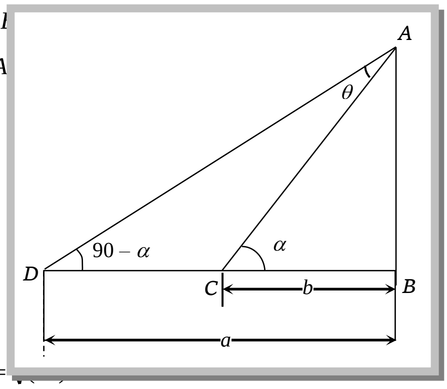
Example: 12 A tower is situated on horizontal plane. From two points, the line joining these points passes through the base and which are a and b distance from the base. The angle of elevation of the top are α and $90^o - \alpha$ and θ is that angle which two points joining the line makes at the top, the height of tower will be **[UPSEAT 1999]**

(a) $\frac{a+b}{a-b}$ (b) $\frac{a-b}{a+b}$ (c) $\sqrt{ab}$ (d) $(ab)^{\frac{1}{3}}$

Solution: (c) Let there are two points C and D on horizontal line passing from point B of the base of the tower AB. The distance of these points are b and a from B respectively *i.e.*, I

$\therefore$ Line CD, on the top of tower A subtends an angle θ , hence $\angle CA$

According to question, on point C and D, the elevation of top are α and $90^o - \alpha$.



$\therefore$ $\angle BCA = \alpha$ and $\angle BDA = 90^o - \alpha$

In ΔABC, $AB = BC \tan\alpha = b\tan\alpha$ ....(i)

and in ΔABD, $AB = BD\tan(90^o - \alpha) = a\cot\alpha$ ....(ii)

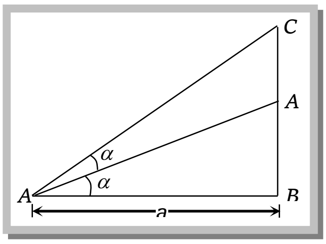
Multiplying equation (i) and (ii) $(AB)^2 = (b\tan\alpha)(a\cot\alpha) = ab, \Rightarrow AB =$

Example: 13 A tower of height b subtends an angle at a point O on the level of the foot of the tower and at a distance a from the foot of the tower. If a pole mounted on the tower also subtends an equal angle at O, the height of the pole is **[MP PET 1993]**

(a) $b\left(\frac{a^2-b^2}{a^2+b^2}\right)$ (b) $b\left(\frac{a^2+b^2}{a^2-b^2}\right)$ (c) $a\left(\frac{a^2-b^2}{a^2+b^2}\right)$ (d) $a\left(\frac{a^2+b^2}{a^2-b^2}\right)$

Solution: (b) Let AB is tower and AC is pole of height h.

From $\Delta ABO, \frac{b}{a} = \tan\alpha$ ......(i)



From ΔCBO, $\frac{b+h}{a} = \tan 2\alpha$ or $\frac{b+h}{a} = \frac{2\tan\alpha}{1-\tan^2\alpha}$

or $b+h = \frac{2a\frac{b}{a}}{1-\frac{b^2}{a^2}}$ (Put value of $\tan\alpha$ from (i))

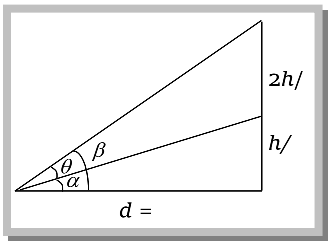
or $h = \frac{b(a^2+b^2)}{a^2-b^2}$.

Remember the result $h = \frac{b(a^2+b^2)}{a^2-b^2}$ in which $b =$ height of tower, $h =$ height of pole, a = distance of observation point from the tower.

Example: 14 A vertical pole consists of two parts, the lower part being one third of the whole. At a point in the horizontal plane through the base of the pole and distance 20 *metres* from it, the upper part of the pole subtends an angle whose tangent is $\frac{1}{2}$. The possible heights of the pole are

(a) 20m and $20\sqrt{3}$ m (b) 20 m and 60 m (c) 16 m and 48 m (d) None of these

Solution: (b) $\frac{H}{3}\cot\alpha = d$ and $H\cot\beta = d$ or $\frac{H}{3d} = \tan\alpha$ and $\frac{H}{d} = \tan\beta$



$\Rightarrow \tan(\beta-\alpha) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{\frac{H}{d}-\frac{H}{3d}}{1+\frac{H^2}{3d^2}} \Rightarrow 1+\frac{H^2}{3d^2} = \frac{4H}{d}$

$\Rightarrow H^2 - 4dH + 3d^2 = 0$

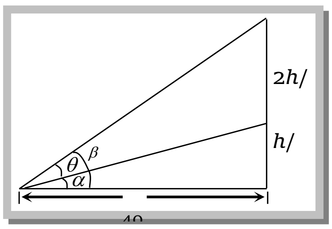
$\Rightarrow H^2 - 80H + 3(400) = 0 \Rightarrow H = 20$ or 60 m.

Example: 15 A vertical pole (more than 100 m high) consists of two portions the lower being $\frac{1}{3}^{rd}$ of the whole. If the upper portion subtends an angle $\tan^{-1}\frac{1}{2}$ at a point in a horizontal plane through the foot of the pole and distance 40 ft. from it, then the height of the pole is

(a) 100 ft. (b) 120 ft. (c) 150 ft. (d) None of these

Solution: (b) Obviously from figure, $\tan\alpha = \frac{h}{120}$ ....(i)

$\tan\beta = \frac{3h}{120}$, ...(ii)



Therefore, $\tan\theta = \tan(\beta-\alpha)$

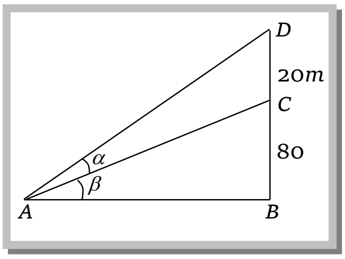
$\Rightarrow \frac{1}{2} = \frac{\frac{3h}{120}-\frac{h}{120}}{1+\frac{3h^2}{14400}} \Rightarrow h = 120, 40$

But $h=40$ can not be taken according to the condition, therefore $h=120\ ft.$.

Example: 16 20 *metre* high flag pole is fixed on a 80 *metre* high pillar, 50 *metre* away from it, on a point on the base of pillar the flag pole makes an angle α, then the value of $\tan\alpha$ is

(a) $\frac{2}{11}$ (b) $\frac{2}{21}$ (c) $\frac{21}{2}$ (d) $\frac{21}{4}$

Solution: (b) Let $\angle BAC=\beta$ $\therefore \tan\beta=\frac{80}{50}$



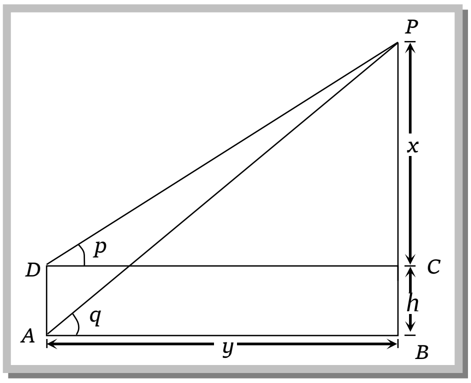
Now $\tan(\alpha+\beta)=\frac{100}{50}$

$\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha.\tan\beta}=2 \Rightarrow \frac{\tan\alpha+\frac{8}{5}}{1-\frac{8}{5}\tan\alpha}=2 \Rightarrow \tan\alpha=\frac{2}{21}$.

Example: 17 The top of a hill observed from the top and bottom of a building of height h is at the angle of elevation p and q respectively. The height of the hill is **[UPSEAT 2001]**

(a) $\frac{h\cot q}{\cot q-\cot p}$ (b) $\frac{h\cot p}{\cot p-\cot q}$ (c) $\frac{h\tan p}{\tan p-\tan q}$ (d) None of these

Solution: (b) Let AD be the building of height h and BP be the hill, then



$\tan q=\frac{h+x}{y}$ and $\tan p=\frac{x}{y}$

$\Rightarrow \tan q=\frac{h+x}{x\cot p} \Rightarrow x\cot p=(h+x)\cot q$

$\Rightarrow x=\frac{h\cot q}{\cot p-\cot q} \Rightarrow h+x=\frac{h\cot p}{\cot p-\cot q}$

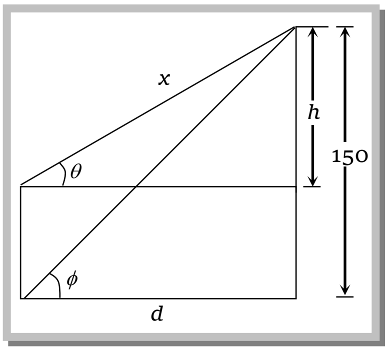
Example: 18 The angular depressions of the top and foot of a chimney as seen from the top of a second chimney, which is 150 m high and standing on the same level as the first are θ and ϕ respectively, then the distance between their tops when $\tan\theta=\frac{4}{3}$ and $\tan\phi=\frac{5}{2}$ is

(a) $\frac{150}{\sqrt{3}}$ *metres* (b) $100\sqrt{3}$ *metres* (c) 150 *metres* (d) 100 *metres*

Solution: (d) $d=150\cot\phi=60m$

Also, $h=60\tan\theta=80m$

Hence, $x=\sqrt{80^2+60^2}=100\ m$.



Example: 19 The angle of elevation of a cliff at a point A on the ground and a point B, 100 m vertically at A are α and β respectively. The height of the cliff is

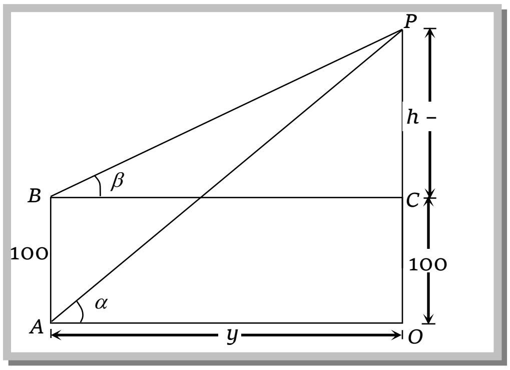
(a) $\frac{100 \cot \alpha}{\cot \alpha - \cot \beta}$ (b) $\frac{100 \cot \beta}{\cot \alpha - \cot \beta}$ (c) $\frac{100 \cot \beta}{\cot \beta - \cot \alpha}$ (d) $\frac{100 \cot \beta}{\cot \beta + \cot \alpha}$

Solution: (c) If $OP = h$, then $CP = h - 100$

Now, equate the values of OA and BC

$h \cot \alpha = (h - 100) \cot \beta$

$\therefore \ h = \frac{100 \cot \beta}{\cot \beta - \cot \alpha}$.



Example: 20 For a man, the angle of elevation of the highest point of the temple situated east of him is 60^o. On walking 240 *metres* to north, the angle of elevation is reduced to 30^o, then the height of the temple is

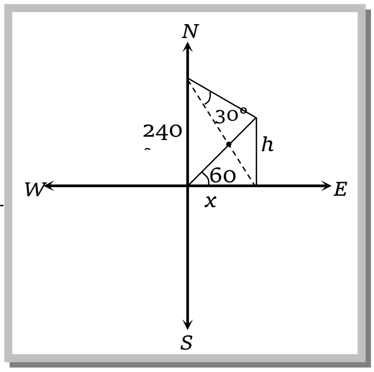
(a) $60\sqrt{6}m$ (b) $60m$ (c) $50\sqrt{3}\ m$ (d) $30\sqrt{6}m$

Solution: (a) Total distance from temple = $\sqrt{x^2 + (240)^2}$

where $x = \frac{h}{\tan 60^o} = \frac{h}{\sqrt{3}}$

So distance = $\sqrt{\frac{h^2}{3} + (240)^2}$, but $\frac{h}{\sqrt{\frac{h^2}{3} + (240)^2}} = \frac{1}{\sqrt{3}} \Rightarrow \frac{h^2}{\frac{h^2}{3} + (240)^2} = \frac{1}{3}$

After solving $h = 60\sqrt{6}$ *metre.*



Example: 21 Two men are on the opposite side of a tower. They measure the angles of elevation of the top of the tower 45^o and 30^o respectively. If the height of the tower is 40*m*, find the distance between the men

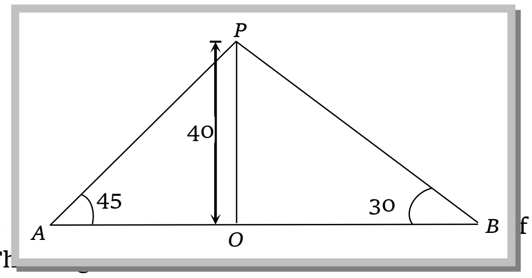
[Karnataka CET 1998]

(a) 40 *m* (b) $40\sqrt{3}m$ (c) 68.280 *m* (d) 109.28 *m*

Solution: (d) $OA = \frac{40}{\tan 45^o}$

$OB = \frac{40}{\tan 30^o}$; $AB = OA + OB = 40[1 + \sqrt{3}]$

$= 40[\sqrt{3} + 1] = 40 \times 2.732 = 109.28$ *metre.*



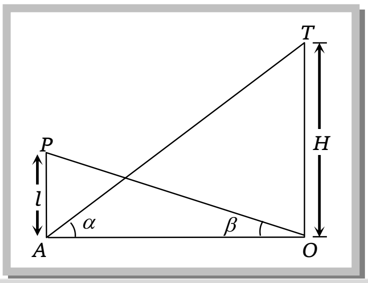
Example: 22 A tower subtends an angle α at a point A in the plane f the foot of the tower at a point l meters just above A is β. Th

(a) $l \tan \beta \cot \alpha$ (b) $l \tan \alpha \cot \beta$ (c) $l \tan \alpha \tan \beta$ (d) $l \cot \alpha \cot \beta$

Solution: (b) $\frac{H}{OA} = \tan \alpha \Rightarrow H = OA \tan \alpha$ ....(i)

$\frac{l}{OA} = \tan \beta \Rightarrow OA = l \cot \beta$ ....(ii)

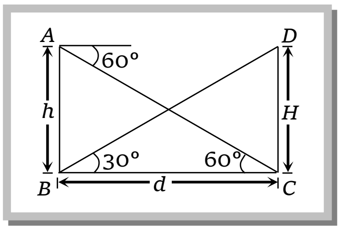
From (i) and (ii) $H = l \tan \alpha \cot \beta$.



Example: 23 A tower subtends an angle of 30° at a point distant d from the foot of the tower and on the same level as the foot of the tower. At a second point h vertically above the first, the depression of the foot of the tower is 60°. The height of the tower is

(a) $\frac{h}{3}$ (b) $\frac{h}{3d}$ (c) $3h$ (d) $\frac{3h}{d}$

Solution: (a) Let CD is tower



From ΔBCD, $\frac{H}{d} = \tan 30^o$ ....(i)

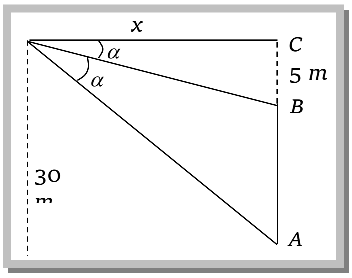
and from ΔABD, $\frac{h}{d} = \tan 60^o$ ....(ii)

Divide equation (ii) from equation (i), we have $\frac{\frac{H}{d}}{\frac{h}{d}} = \frac{\tan 30^o}{\tan 60^o} \Rightarrow H = \frac{h}{3}$.

Example: 24 A flag-staff of 5m high stands on a building of 25 m high. At an observer at a height of 30m. The flag-staff and the building subtend equal angles. The distance of the observer from the top of the flag-staff is **[EAMCET 1993]**

(a) $\frac{5\sqrt{3}}{2}$ (b) $5\sqrt{\frac{3}{2}}$ (c) $5\sqrt{\frac{2}{3}}$ (d) None of these

Solution: (b) We have, $\tan\alpha = \frac{5}{x}$ and $\tan 2\alpha = \frac{30}{x}$



$\therefore \tan 2\alpha = \frac{30}{5\cot\alpha} \Rightarrow \tan 2\alpha = 6\tan\alpha$

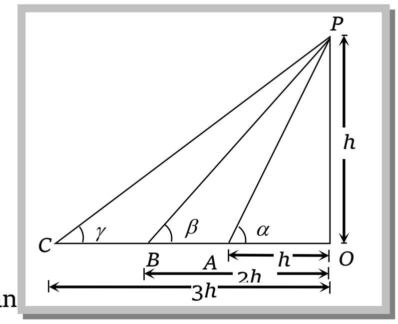
$\Rightarrow 3 - 3\tan^2\alpha = 1 \Rightarrow \tan\alpha = \sqrt{\frac{2}{3}}$

$\therefore x = 5\cot\alpha = 5\sqrt{\frac{3}{2}}$.

Example: 25 The length of the shadows of a vertical pole of height h, thrown by the sun's ray at three different moments are h, $2h$ and $3h$. The sum of the angles of elevation of the rays at these three moments is equal to **[MP PET 2000]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Solution: (a) $\tan\alpha = \frac{h}{h} = 1 \Rightarrow \alpha = 45^o$



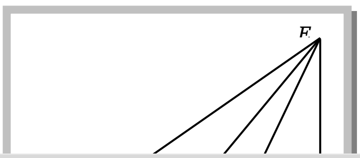
$\tan\beta = \frac{h}{2h} \Rightarrow \beta = \tan^{-1}\left(\frac{1}{2}\right)$; $\tan\gamma = \frac{h}{3h} \Rightarrow \gamma = \tan^{-1}\left(\frac{1}{3}\right)$

$\therefore \alpha + \beta + \gamma = 45^o + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45^o + 45^o = \frac{\pi}{2}$.

Example: 26 A tower subtends angles $\alpha, 2\alpha, 3\alpha$ respectively at points A, B an line through the foot of the tower. Then $\frac{AB}{BC} =$

(a) $\frac{\sin 3\alpha}{\sin 2\alpha}$ (b) $1 + 2\cos 2\alpha$ (c) $2 + \cos 3\alpha$ (d) $\frac{\sin 2\alpha}{\sin\alpha}$

Solution: (b) From sine rule



$\Rightarrow \dfrac{BE}{\sin(180^o - 3\alpha)} = \dfrac{BC}{\sin\alpha}$

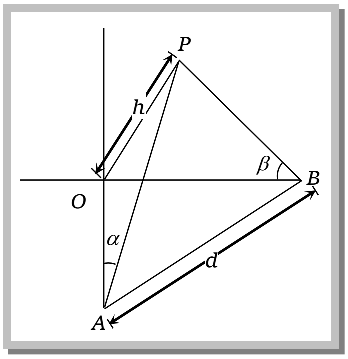
$\Rightarrow \dfrac{AB}{\sin 3\alpha} = \dfrac{BC}{\sin\alpha}$ (Since $BE = AB$)

$\Rightarrow \dfrac{AB}{BC} = \dfrac{\sin 3\alpha}{\sin\alpha} = 3 - 4\sin^2\alpha \Rightarrow \dfrac{AB}{BC} = 3 - 2(1-\cos 2\alpha) \Rightarrow \dfrac{AB}{BC} = 1 + 2\cos 2\alpha$.

Example: 27 The angle of elevation of the top of a tower from a point A due south of the tower is α and from a point B due east of the tower is β. If $AB = d$, then the height of the tower is

(a) $\dfrac{d}{\sqrt{\tan^2\alpha - \tan^2\beta}}$ (b) $\dfrac{d}{\sqrt{\tan^2\alpha + \tan^2\beta}}$

(c) $\dfrac{d}{\sqrt{\cot^2\alpha + \cot^2\beta}}$ (d) $\dfrac{d}{\sqrt{\cot^2\alpha - \tan^2\beta}}$



Solution: (c) $OB = h\cot\beta$

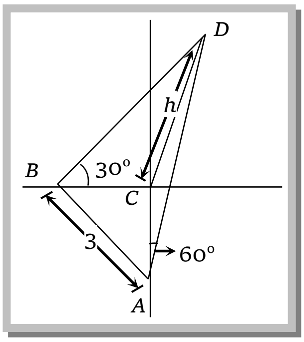
$OA = h\cot\alpha$

$\Rightarrow h^2 = \dfrac{d^2}{\cot^2\beta + \cot^2\alpha} \Rightarrow h = \dfrac{d}{\sqrt{\cot^2\beta + \cot^2\alpha}}$.

Example: 28 The angular elevation of a tower CD at a point A due south of it is 60° and at a point B due west of A, the elevation is 30^o. If $AB = 3$ km, the height of the tower is

(a) $2\sqrt{3}\ km$ (b) $2\sqrt{6}\ km$

(c) $\dfrac{3\sqrt{3}}{2}\ km$ (d) $\dfrac{3\sqrt{6}}{4}\ km$



Solution: (d) In ΔCBD, $\tan 30^o = \dfrac{h}{BC} \Rightarrow BC = \sqrt{3}h$,

In ΔACD, $\tan 60^o = \dfrac{h}{AC} \Rightarrow AC = \dfrac{h}{\sqrt{3}}$

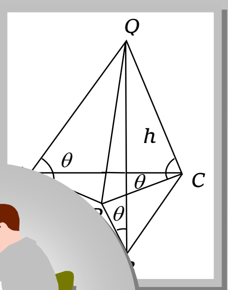
Now, $AB^2 = AC^2 + CB^2$, $3^2 = 3h^2 + \dfrac{h^2}{3}$, $27 = 10h^2$

$h^2 = \dfrac{27}{10} \Rightarrow h = \dfrac{3\sqrt{3}}{\sqrt{10}} = \dfrac{3\sqrt{6}}{\sqrt{20}} \simeq \dfrac{3\sqrt{6}}{4}\ km.$

Example: 29 A pole stands vertically inside a triangular park ΔABC. If the angle of elevation of the top of the pole from each corner of the park is same, then in Δ ABC the foot of the pole is at the

(a) Centroid (b) Circumcentre

(c) Incentre (d) Orthocentre



Solution: (b) Let PQ be the pole, since the angle of Q from each of the points A, B, C is the sa

$\therefore$ $PA = PB = PC = h\cot\theta$

Since P is equidistant from A, B, C.

$\therefore$ P is circumcentre of ΔABC.

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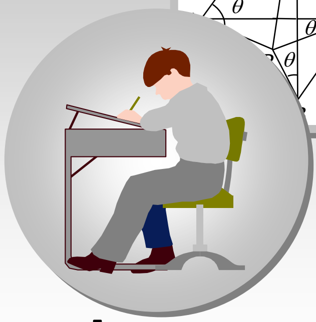

Assignment

Height and Distance

Basic Level

1. The angle of elevation of the sun, when the shadow of the pole is $\sqrt{3}$ times the height of the pole, is

[MP PET 1991, 1996; SCRA 1999]

(a) 60^o (b) 30^o (c) 45^o (d) 15^o

2. Some portion of a 20 *meters* long tree is broken by the wind and the top struck the ground at an angle of 30°. The height of the point where the tree is broken is

(a) 10*m* (b) $(2\sqrt{3}-3)20m$ (c) $\frac{20}{3}m$ (d) None of these

3. A tree is broken by wind, its upper part touches the ground at a point 10 meters from the foot of the tree and makes an angle of 45° with the ground. The total length of tree is

(a) 15 *metres* (b) 20 *metres* (c) $10(1+\sqrt{2})$ metres (d) $10\left(1+\frac{\sqrt{3}}{2}\right)$ metres

4. From the roof of a 15 *metre* high house the angle of elevation of a point located 15 *metre* distant to the base of the house is

[MP PET 1988]

(a) 45^o (b) 30^o (c) 60^o (d) 90^o

5. The angle of depression of a ship from the top of a tower 30 *metre* high is 60°, then the distance of ship from the base of tower is

[MP PET 1988]

(a) $30m$ (b) $30\sqrt{3}m$ (c) $10\sqrt{3}m$ (d) $10m$

6. If a flagstaff of 6 *metres* high placed on the top of a tower throws a shadow of $2\sqrt{3}$ *metres* along the ground, then the angle (in degrees) that the sun makes with the ground is

(a) 60^o (b) 80^o (c) 75^o (d) None of these

7. The angle of depression of a point situated at a distance of 70 *metres* from the base of a tower is 45°. The height of the tower is

[MP PET 1997]

(a) 70*m* (b) $70\sqrt{2}m$ (c) $\frac{70}{\sqrt{2}}m$ (d) $35m$

8. The tops of two poles of height 20*m* and 14*m* are connected by a wire. If the wire makes an angle 30° with the horizontal, then the length of the wire is

(a) 12 *m* (b) 10*m* (c) 8*m* (d) None of these

9. The angle of elevation of the top of a tower at a point on the ground is 30^o. If on walking 20 *metres* toward the tower, the angle of elevation becomes 60^o, then the height of the tower is

(a) 10 *metre* (b) $\frac{10}{\sqrt{3}}$ *metre* (c) $10\sqrt{3}$ *metres* (d) None of these

10. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°. When he retirs 40 *meters* from the bank, he finds the angle to be 30°. The breadth of the river is

(a) 20*m* (b) 40*m* (c) 30*m* (d) 60*m*

11. A person walking along a straight road towards a hill observes at two points distance $\sqrt{3}$ kms., the angles of elevation of the hill to be 30° and 60^o. The height of the hill is

(a) 3/2 *km* (b) $\sqrt{2/3}$ *km* (c) $\sqrt{2}+1/2$ *km* (d) $\sqrt{3}$ *kms*

12. An observer in a boat finds that the angle of elevation of a tower standing on the top of a cliff is 60° and that of the top of cliff is 30°. If the height of the tower be 60 *meters*, then the height of the cliff is

(a) 30 *m* (b) $60\sqrt{3}m$ (c) $20\sqrt{3}m$ (d) None of these

13. The upper 3/4th portion of a vertical pole subtends an angle $\tan^{-1} 3/5$ at a point in the horizontal plane through its foot and at a distance 40 *m* from the foot. A possible height of the vertical pole is

(a) 20 *m* (b) 40 *m* (c) 60 *m* (d) 80 *m*

14. *AB* is a vertical tower. The point *A* is on the ground and *C* is the middle point of *AB*. The part *CB* subtend an angle α at a point P on the ground. If $AP = n\,AB$. then the correct relation is

(a) $n = (n^2+1)\tan\alpha$ (b) $n = (2n^2-1)\tan\alpha$ (c) $n^2 = (2n^2+1)\tan\alpha$ (d) $n = (2n^2+1)\tan\alpha$

15. From an aeroplane vertically over a straight horizontally road, the angles of depression of two consecutive mile stones on opposite sides of the aeroplane are observed to be α and β, then the height in miles of aeroplane above the road is **[MNR 1986; UPSEAT 1999]**

(a) $\frac{\tan\alpha.\tan\beta}{\cot\alpha+\cot\beta}$ (b) $\frac{\tan\alpha+\tan\beta}{\tan\alpha.\tan\beta}$ (c) $\frac{\cot\alpha+\cot\beta}{\tan\alpha.\tan\beta}$ (d) $\frac{\tan\alpha.\tan\beta}{\tan\alpha+\tan\beta}$

16. The angle of elevation of the top of a tower from the top and bottom of a building of height *a* are 30° and 45° respectively. If the tower and the building stand at the same level, the height of the tower is

(a) $a\sqrt{3}$ (b) $a\sqrt{3}-1$ (c) $a\left(3+\frac{\sqrt{3}}{2}\right)$ (d) $a\sqrt{3}+1$

17. From the bottom of a pole of height *h*, the angle of elevation of the top of a tower is α and the pole subtends angle β at the top of the tower. The height of the tower is

(a) $\frac{h\tan(\alpha-\beta)}{\tan(\alpha-\beta)-\tan\alpha}$ (b) $\frac{h\cot(\alpha-\beta)}{\cot(\alpha-\beta)-\cot\alpha}$ (c) $\frac{\cot(\alpha-\beta)}{\cot(\alpha-\beta)-\cot\alpha}$ (d) None of these

18. From the bottom and top of a house *h meter* high, the angles of elevation of the top of a tower are α andβ. The height of the tower is **[EAMCET 1989]**

(a) $\frac{h\sin\beta}{\cos\beta-\sin\alpha}$ (b) $\frac{h\cos\beta}{\cos\beta-\cos\alpha}$ (c) $\frac{h\tan\beta}{\tan\beta-\tan\alpha}$ (d) $\frac{h\cot\beta}{\cot\beta-\cot\alpha}$

19. If the angles of elevation of two towers from the middle point of the line joining their feet be 60° and 30° respectively, then the ratio of their heights is

(a) 2 : 1 (b) $1:\sqrt{2}$ (c) 3:1 (d) $1:\sqrt{3}$

20. A ladder rests against a wall making an angle α with the horizontal. The foot of the ladder is pulled away from the wall through a distance *x*, so that it slides a distance *y* down the wall making an angle β with the horizontal. The correct relation is **[IIT 1985]**

(a) $x = y\tan\frac{\alpha+\beta}{2}$ (b) $y = x\tan\frac{\alpha+\beta}{2}$ (c) $x = y\tan(\alpha+\beta)$ (d) $y = x\tan(\alpha+\beta)$

21. The length of the shadow of a pole inclined at 10° to the vertical towards the sun is 2.05 meters, when the elevation of the sun is 38°. The length of the pole is

(a) $\frac{2.05\sin 38^o}{\sin 42^o}$ (b) $\frac{2.05\sin 42^o}{\sin 38^o}$ (c) $\frac{2.05\cos 38^o}{\cos 42^o}$ (d) None of these

22. An aeroplane flying horizontally 1 *km* above the ground is observed at an elevation of 60^o and after 10 seconds the elevation is observed to be 30^o. The uniform speed of the aeroplane in km/h is

(a) 240 (b) $240\sqrt{3}$ (c) $60\sqrt{3}$ (d) None of these

23. From a point *a* meter above a lake the angle of elevation of a cloud is α and the angle of depression of its reflection is β. The height of the cloud is **[Roorkee 1983; EAMCET 1983, 1985]**

(a) $\frac{a\sin(\alpha+\beta)}{\sin(\alpha-\beta)}m$ (b) $\frac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}m$ (c) $\frac{a\sin(\beta-\alpha)}{\sin(\alpha+\beta)}m$ (d) None of these

24. A house subtends a right angle at the window of an opposite house and the angle of elevation of the window; from the bottom of the first house is 60°. If the distance between the two houses be 6 meters, then the height of the first house is **[MNR 1978]**

(a) $6\sqrt{3}m$ (b) $8\sqrt{3}m$ (c) $4\sqrt{3}m$ (d) None of these

25. The angle of elevation of a stationary cloud from a point 2500 *m* above a lake is 15° and the angle of depression of its reflection in the lake is 45°. The height of cloud above the lake level is

(a) $2500\sqrt{3}m$ (b) 2500 *m* (c) $500\sqrt{3}\,m$ (d) None of these

Advance Level

26. *AB* is a vertical pole resting at the end *A* on the level ground. *P* is a point on the level ground such that $AP = 3AB$. If *C* is the mid-point of *AB* and *CB* subtends an angle β at *P*, the value of $\tan\beta$ is

(a) $\frac{18}{19}$ (b) $\frac{3}{19}$ (c) $\frac{1}{6}$ (d) None of these

27. The angle of elevation of the top of an unfinished tower at a point distant 120m from its base is 45°. If the elevation of the top at the same point is to be 60°, the tower must be raised to a height

(a) $120(\sqrt{3}+1)\,m$ (b) $120(\sqrt{3}-1)\,m$ (c) $60(\sqrt{3}+1)\,m$ (d) None of these

28. An aeroplane flying at a height of 300 *metres* above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from the same point on the ground are 60° and 45° respectively. The height of the lower plane from the ground (in *metres*) is

(a) $100\sqrt{3}$ (b) $\frac{100}{\sqrt{3}}$ (c) 50 (d) $150(\sqrt{3}+1)$

29. At a point on a level plane a tower subtends an angle θ and a flag staff *a* ft. in length at the top of the tower subtends an angle ϕ. The height of the tower is

(a) $\frac{a\sin\theta\cos\phi}{\cos(\theta+\phi)}$ (b) $\frac{a\sin\phi\cos(\theta+\phi)}{\sin\theta}$ (c) $\frac{a\cos(\theta+\phi)}{\sin\theta\sin\phi}$ (d) None of these

30. From the top of a cliff of height *a* the angle of depression of the foot of a certain tower is found to be double the angle of elevation of the top of the tower of height *h*. If θ be the angle of elevation then its value is

(a) $\cos^{-1}\sqrt{\frac{2h}{a}}$ (b) $\sin^{-1}\sqrt{\frac{2h}{a}}$ (c) $\sin^{-1}\sqrt{\frac{a}{2-h}}$ (d) $\tan^{-1}\sqrt{3-\frac{2h}{a}}$

31. A spherical balloon of radius *r* subtends an angle α at the eye of an observer. If the angle of elevation of the centre of the balloon be β, the height of the centre of the balloon is

(a) $r\text{cosec}\left(\frac{\alpha}{2}\right)\sin\beta$ (b) $r\,\text{cosec}\ \alpha\sin\left(\frac{\beta}{2}\right)$ (c) $r\sin\left(\frac{\alpha}{2}\right)\text{cosec}\ \beta$ (d) $r\sin\alpha\ \text{cosec}\left(\frac{\beta}{2}\right)$

32. A stationary balloon is observed from three points *A*, *B* and *C* on the plane ground and is found that its angle of elevation from each of these points is α. If ∠ABC = β and *AC* = *b*, the height of the balloon is

(a) $\frac{b}{2\sin\beta\cot\alpha}$ (b) $\frac{2b}{\sin\beta\cot\alpha}$ (c) $\frac{b}{2\sin\alpha\cot\beta}$ (d) $\frac{2b}{\sin\alpha\cot\beta}$

33. The angle of elevation of the top of the tower observed from each of the three points *A, B, C* on the ground, forming a triangle is the same angle α. If *R* is the circum-radius of the triangle *ABC*, then the height of the tower is **[EAMCET 1994]**

(a) $R\sin\alpha$ (b) $R\cos\alpha$ (c) $R\cot\alpha$ (d) $R\tan\alpha$

34. A balloon is observed simultaneously from three points *A*, *B* and *C* on a straight road directly under it. The angular elevation at *B* is twice and at *C* is thrice that of *A*. If the distance between *A* and *B* is 200 metres and the distance between *B* and *C* is 100 metres, then the height of balloon is given by

(a) 50 *metres* (b) $50\sqrt{3}$ *metres* (c) $50\sqrt{2}$ *metres* (d) None of these

35. Three poles whose feet *A, B, C* lie on a circle subtend angles α, β, γ respectively at the centre of the circle. If the height of the poles are in A.P., then $\cot\alpha, \cot\beta, \cot\gamma$ are in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

36. *PQ* is a vertical tower. *P* is the foot, *Q* the top of the tower, *A*, *B*, *C* are three points in the horizontal plane through *P*. The angles of elevation of Q from *A*, *B*, *C* are equal and each is equal to θ. The sides of the triangle *ABC* are a, *b*, *c* and the area of the triangle *ABC* is Δ. The height of the tower is

(a) $\frac{abc\tan\theta}{4\Delta}$ (b) $(abc)\cot\frac{\theta}{4\Delta}$ (c) $(abc)\sin\frac{\theta}{4\Delta}$ (d) None of these

37. A tower AB leans towards west making an angle α with the vertical. The angular elevation of B, the topmost point of the tower is β as observed from a point C due west of A at a distance d from A. If the angular elevation of B from a point D due east of C at a distance $2d$ from C is γ, then $2\tan\alpha$ can be given as

(a) $3\cot\beta - 2\cot\gamma$ (b) $3\cot\gamma - 2\cot\beta$ (c) $3\cot\beta - \cot\gamma$ (d) $\cot\beta - 3\cot\gamma$

38. ABC is a triangular park with AB=AC=100 m. A clock tower is situated at the mid of BC. The angles of elevation of the top of the tower at A and B are $\cot^{-1}3.2$ and $\operatorname{cosec}^{-1}2.6$ respecti f the tower is

(a) 50 m (b) 25 m (c) 40 m hese

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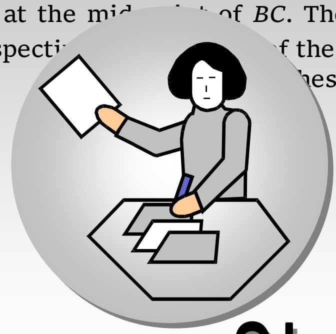

Answer Sheet

Heights and Distances ***Assignment (Basic & Advance Level)***

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
b	c	c	a	c	a	a	a	c	a	a	a	b	d	d	c	b	d	c	a
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38		
a	b	b	b	a	b	b	a	b	d	a	a	d	d	c	a	c	b		